

Non-linear stability in photogravitational non-planar restricted three body problem with oblate smaller primary

B. Ishwar and J. P. Sharma

University Department of Mathematics, B.R.A. Bihar university, Muzaffarpur-842001

ishwar_bhola@hotmail.com

ABSTRACT

We have discussed non-linear stability in photogravitational non-planar restricted three body problem with oblate smaller primary. By photogravitational we mean that both primaries are radiating. We normalised the Hamiltonian using Lie transform as in Coppola and Rand (1989). We transformed the system into Birkhoff's normal form. Lie transforms reduce the system to an equivalent simpler system which is immediately solvable. Applying Arnolds theorem, we have found non-linear stability criteria. We conclude that L_6 is stable. We plotted graphs for (ω_1, D_2) . They are rectangular hyperbola.

Subject headings: Non-linear stability:Photogravitational:Non-planar:Oblate primary:RTBP

1. Introduction

G. Hori(1966,1967) applied a theorem by Lie in canonical transformation to the theory of general perturbations. Theorem is applicable to such cases where the undisturbed portion of Hamiltonian depends on angular variable as well as momentum variables. A. Deprit(1969) introduced the concept of Lie series to the cases where the generating function itself depends explicitly on the small parameter. Lie transforms define naturally a class of canonical mappings in the form of power series in the small parameter. They reviewed how a Lie series defines a canonical mapping as a formal power series of a small parameter ϵ , provided the generating function itself does not depend upon ϵ . This restriction is overcome by introducing Lie transform. They showed that how they naturally define the canonical transformations contemplated by Von Zeipel's method. Canonical mappings defined by Lie transforms as formal power series of a small parameter constitute the natural ingredient of Transformation Theory applied to Hamiltonian systems.

Orbital stability of quasi-periodic motions in multidimensional Hamiltonian systems was studied by Sokolskii(1978). With some applications to the Birkhoff’s normal form along with its generalized form by K. R. Meyer, the restricted problem of three bodies near L_4 , the Birkhoff’s normalization procedure, and the singular perturbation, of Hamiltonian systems have been discussed by Liu(1985). K.R. Meyer and D.S.Schmidt(1986) established the full stability of Lagrange equilibrium point in the planar restricted three body problem even in the case when $\mu = \mu_c$. Hamiltonian is normalised up to order six and then KAM theory is applied. This establishes the stability of the equilibrium in degenerate case. A.P. Markeev(1966) and K.T. Alfriend(1970,1971) have shown that L_4 is unstable when the mass ratio is equal to μ_2 or μ_3 . The Lie transform method is an efficient perturbation scheme which explicitly generates the functional form of the reduced Hamiltonian under an implicitly defined canonical periodic near identity - transformation.

Coppola & Rand(1989) applied a method of Lie Transforms, a perturbation method for differential equations to a general class of Hamiltonian systems using computer algebra. They developed explicit formulas for transforming the system into Birkhoff normal form. They formed explicit nonlinear stability criteria solely in terms of H for systems where the linear stability is inconclusive. They applied these results to the non-linear stability of L_4 in the Circular RTBP. At L_4 , Arnold’s theorem(1961) must be used since a Lyapunov function cannot be found. They confirmed the previous computations of Deprit and Deprit Bartholome (1967), Meyer and Schmidt (1986). Algorithms of linear and nonlinear normalization of a Hamiltonian system near an equilibrium point were described by Maciejewski, and Gozdziwski(1991).

A. Jorba(1997) described the effective computation of normal forms, centre manifolds and first integrals in Hamiltonian Mechanics. These kind of conclusions are very useful. They allow, for example, to give explicit estimates on the diffusion time or to compute invariant tori. Their approach presented here is based on using algebraic manipulation for the formal series but taking numerical co-efficients for them.

J. Palacian & P. Yanguas (2000) described the reduction of perturbed Hamiltonian systems. They used a technique based on Lie-transformations. By extending an integral of the unperturbed part to the whole transformed system up to a certain order of approximation, the number of degrees of freedom of such a system is reduced, under certain conditions. The idea of reducing a perturbed system is valid not only for Hamiltonians, but also for any system of differential equations. Recently there has been a resurgence of this subject. The problem of building formal integrals for Hamiltonian systems has received a wide treatment in the last forty five years. The results have been applied in fields such as Molecular Physics or Astrodynamics. For example, in galactic models of three degrees of freedom, the search for

third integral becomes very important to analyze the onset of Chaos. Their approach consists in generalising the concept of normal forms by selecting a function $G(x)$ and proposing thereafter a symplectic change of variables.

The nonlinear stability of triangular equilibrium points was studied by Kushvah et al (2007) in the generalised photogravitational restricted three body problem with Poynting-Robertson drag. They have performed first and second order normalization of the Hamiltonian of the problem and applied KAM theorem to examine the condition of non-linear stability. After computation they have found three critical mass ratios and concluded that triangular points are stable in the nonlinear sense except three critical mass ratios at which KAM theorem fails.

Hence, we thought to examine the Non-linear stability of L_6 , equilibrium point of Non-planar photogravitational restricted three body problem with oblate smaller primary. We examined the linear stability of above problem in Shankaran(2011) where q_1 is radiation pressure of bigger primary and q_2 that of smaller. We have found that L_6 is unstable due to positive real part in complex roots. Now, we proceed to normalize the Hamiltonian of the problem as in Coppola and Rand (1989) using Lie transform. We transformed the system into Birkhoff's normal form. Using Arnold's theorem, we have found non-linear stability criteria in terms of H . Lie transforms reduce the system to an equivalent simpler system which is immediately solvable. Finally we find that $D_2 \neq 0$. Hence, according to Arnold theorem, we conclude that L_6 is stable. We plotted graphs for (ω_1, D_2) . They are rectangular hyperbola.

2. Computation of D_2

We suppose that q is the radiation co-efficient of bigger primary and Q that of smaller primary in photogravitational non-planar restricted three body problem. We want to apply Arnold's theorem(1961). We applied method of Lie transform using computer algebra as given in research article of Coppola and Rand(1989). We suppose $L_n = \{, w_n\}$ and $S_n = \frac{1}{n} \sum_{m=0}^{n-1} L_{n-m} S_m$, (where $n = 1, 2, 3, \dots$) be the operators. Then we transformed the near-identity transformation (x_m, z_m) to (X_n, Z_m) variables. The Kamiltonian is given by

$$K_n = H_n + \frac{1}{2} \{H_n, w_n\} + \frac{1}{2} \sum_{m=0}^{n-1} [L_{n-m} K_m + m S_{n-m} H_m] \quad n = 2, 3, 4, \dots \quad (1)$$

We choose the generating function w_n in such a way so as to best simplify Kamiltonian and each term to be canceled will be of the form $AX_1^j Y_1^l X_2^r Y_2^s$ where A is a constant. We choose w_n to be a sum of terms, one for each term to be canceled, of the form $w_n = AX_1^j Y_1^l X_2^r Y_2^s$

where $B = \frac{iA_n}{\omega_1(l-j)+\omega_3(s-r)}$ with $n = j + l + r + s - 2$. Here ω_1 and ω_3 are the basic frequencies which are rationally independent and we suppose that the frequency in vertical direction is constant i.e. $\omega_3 = 1$. If $j = l$ and $s = r$ then B will be infinite hence the terms $(X_1Y_1)^j(X_2Y_2)^r$. The D_2 is given as (please see Coppola and Rand(1989))

$$D_2 = -(K2200\omega_3^2 + K1111\omega_1\omega_3 + K0022\omega_1^2) \quad (2)$$

We performed computation using Mathematica and found the following results:

$$\begin{aligned} K0022 &= H0022 + i \left(\frac{1}{\omega_1} H0111 \times H1011 + \frac{1}{\omega_3} (H0012 \times H0021 \right. \\ &\quad \left. + H0003 \times H0030) \right. \\ &\quad \left. + \frac{1}{(\omega_1 - 2\omega_3)} H0120 \times H1002 + \frac{1}{(\omega_1 + 2\omega_3)} H0102 \times H1020 \right); \end{aligned} \quad (3)$$

$$\begin{aligned} K1111 &= H1111 + i \left(\frac{1}{\omega_1} (H1011 \times H1200 \right. \\ &\quad \left. + H0111 \times H2100) + \frac{1}{(\omega_1 - 2\omega_3)} H0120 \times H1002 + \right. \\ &\quad \left. \frac{1}{2\omega_3} (H1101 \times H0021 + H1110 * H0012) + \frac{1}{(2\omega_1 - \omega_3)} H0210 \times H2001 + \right. \\ &\quad \left. \frac{1}{(\omega_1 + 2\omega_3)} H0102 \times H1020 + \frac{1}{(2\omega_1 + \omega_3)} H0201 \times H2010 \right); \end{aligned} \quad (4)$$

$$\begin{aligned} K2200 &= H2200 + i \left(\frac{1}{\omega_3} H1101 \times H1110 + \frac{1}{\omega_1} (H1200 \times H2100 \right. \\ &\quad \left. + H0300 \times H3000) + \frac{1}{(2\omega_1 - \omega_3)} H0210 \times H2001 + \frac{1}{(2\omega_1 + \omega_3)} H0201 \times H2010 \right) \end{aligned} \quad (5)$$

$$K2200 = \frac{(5a_1^2 - 6b_1\omega_1)\omega_3(4\omega_1^2 - \omega_3^2) + 2a_2^2\omega_1(4\omega_1^2 + \omega_1\omega_3 - \omega_3^2)}{16\omega_1^3\omega_3 - 4\omega_1\omega_3^3} \quad (6)$$

$$K1111 = \frac{1}{8} \left(-8b_3 + \frac{12a_1a_3}{\omega_1} + \frac{a_3^2}{\omega_1 - 2\omega_3} + \frac{a_2^2}{2\omega_1 - \omega_3} + \frac{6a_2a_4}{\omega_3} + \frac{a_2^2}{2\omega_1 + \omega_3} + \frac{a_3^2}{\omega_1 + 2\omega_3} \right) \quad (7)$$

$$K0022 = \frac{1}{8} \left(-12b_5 + \frac{10a_4^2}{\omega_3} + a_3^2 \left(\frac{4}{\omega_1} + \frac{2\omega_1}{\omega_1^2 - 4\omega_3^2} \right) \right) \quad (8)$$

$$D_2 = - (K2200\omega_3^2 + K1111\omega_1\omega_3 + K0022\omega_1^2) \quad (9)$$

$$= \frac{1}{4} \left(-3a_2a_4\omega_1 + 6b_5\omega_1^2 - \frac{5a_4^2\omega_1^2}{\omega_3} - 6a_1a_3\omega_3 + 4b_3\omega_1\omega_3 + \right. \\ \left. 6b_1\omega_3^2 - \frac{5a_1^2\omega_3^2}{\omega_1} - \frac{a_3^2\omega_1(3\omega_1^2+\omega_1\omega_3-8\omega_3^2)}{\omega_1^2-4\omega_3^2} + \frac{2a_2^2\omega_3(5\omega_1^2+\omega_1\omega_3-\omega_3^2)}{-4\omega_1^2+\omega_3^2} \right)$$

where

$$a = (1 - \mu) + \frac{6(3)^{1/2}((1-\mu))(1-q)A^{3/2}}{\mu Q};$$

$$c = (3 * A)^{1/2} - \frac{9((1-\mu))qA^2}{\mu Q};$$

$$a_1 = \frac{-q\mu^4 + Q\sqrt{(-1+\mu)^2}\sqrt{\mu^2} - 2Q\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} + Q\sqrt{(-1+\mu)^2}(\mu^2)^{3/2}}{(-1+\mu)^2\sqrt{(-1+\mu)^2}\mu^4} - \\ \left(5 \left(-3q\mu^6 + 2Q\sqrt{(-1+\mu)^2}\sqrt{\mu^2} - 8Q\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} - 8Q\sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} + \right. \right. \\ \left. 2Q\sqrt{(-1+\mu)^2}\mu^4\sqrt{\mu^2} + 12Q\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} \right) A \Big/ \\ \left((-1+\mu)^4\sqrt{(-1+\mu)^2}\mu^6 \right) + \left(24 \left(\sqrt{3}q\mu^5 - \sqrt{3}q^2\mu^5 + \sqrt{3}Q\sqrt{(-1+\mu)^2}\sqrt{\mu^2} - \right. \right. \\ \left. \sqrt{3}qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2} - 3\sqrt{3}Q\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} + \right. \\ \left. 3\sqrt{3}qQ\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} - \sqrt{3}Q\sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} + \right. \\ \left. \sqrt{3}qQ\sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} + 3\sqrt{3}Q\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} - \right. \\ \left. 3\sqrt{3}qQ\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} \right) A^{3/2} \Big/ \left(Q(-1+\mu)^2\sqrt{(-1+\mu)^2}\mu^6 \right) - \\ \left(945 \left(q\mu^8 + Q\sqrt{(-1+\mu)^2}\sqrt{\mu^2} - 6Q\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} - 20Q\sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} + \right. \right. \\ \left. 15Q\sqrt{(-1+\mu)^2}\mu^4\sqrt{\mu^2} - 6Q\sqrt{(-1+\mu)^2}\mu^5\sqrt{\mu^2} + \right. \\ \left. Q\sqrt{(-1+\mu)^2}\mu^6\sqrt{\mu^2} + 15Q\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} \right) A^2 \Big/ \\ \left(8 \left((-1+\mu)^6\sqrt{(-1+\mu)^2}\mu^8 \right) \right) + O[A]^{5/2};$$

$$a_2 = \left(-\frac{6\sqrt{3}q}{((1-\mu)^2)^{5/2}} + \frac{15\sqrt{3}q\mu}{2((1-\mu)^2)^{5/2}} - \frac{3\sqrt{3}q\sqrt{(1-\mu)^2}\mu}{2(1-\mu)^6} - \frac{6\sqrt{3}Q\sqrt{\mu^2}}{\mu^5} \right) \sqrt{A} - \\ \frac{135(\sqrt{3}q)A^{3/2}}{2((-1+\mu)^5\sqrt{(-1+\mu)^2})} + \\ \left(54 \left(-10q\mu^6 + 9q^2\mu^6 + q^2\mu^7 + 10Q\sqrt{(-1+\mu)^2}\sqrt{\mu^2} - 10qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2} - \right. \right. \\ \left. 40Q\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} + 39qQ\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} - \right. \\ \left. 40Q\sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} + 34qQ\sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} + 10Q\sqrt{(-1+\mu)^2} \right. \\ \left. \mu^4\sqrt{\mu^2} - 6qQ\sqrt{(-1+\mu)^2}\mu^4\sqrt{\mu^2} - qQ\sqrt{(-1+\mu)^2}\mu^5\sqrt{\mu^2} + \right. \\ \left. 60Q\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} - 56qQ\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} \right) A^2 \Big/$$

$$\left(Q(-1 + \mu)^3 \sqrt{(-1 + \mu)^2} \mu^7 \right) + O[A]^{5/2};$$

$$\begin{aligned} a_3 = & \left(\frac{3q(1-\mu)}{2((1-\mu)^2)^{5/2}} - \frac{3q(1-\mu)\mu}{2((1-\mu)^2)^{5/2}} - \frac{3Q\sqrt{\mu^2}}{2\mu^4} \right) + \\ & \left(-\frac{45q(1-\mu)}{2((1-\mu)^2)^{7/2}} - \frac{45q}{4(1-\mu)((1-\mu)^2)^{5/2}} + \frac{45q(1-\mu)\mu}{2((1-\mu)^2)^{7/2}} + \frac{45q\mu}{4(1-\mu)((1-\mu)^2)^{5/2}} + \right. \\ & \left. \frac{45Q\sqrt{\mu^2}}{2\mu^6} \right) A + \left(\frac{36\sqrt{3}(1-q)q(1-\mu)}{Q((1-\mu)^2)^{5/2}} - \frac{36\sqrt{3}(1-q)q(1-\mu)}{Q((1-\mu)^2)^{5/2}\mu} - \right. \\ & \left. \frac{36\sqrt{3}\sqrt{\mu^2}}{\mu^6} + \frac{36\sqrt{3}q\sqrt{\mu^2}}{\mu^6} + \frac{36\sqrt{3}\sqrt{\mu^2}}{\mu^5} - \frac{36\sqrt{3}q\sqrt{\mu^2}}{\mu^5} \right) A^{3/2} + \\ & \left(\frac{945q}{4(1-\mu)((1-\mu)^2)^{7/2}} + \frac{945q}{16(1-\mu)^3((1-\mu)^2)^{5/2}} - \frac{945q\mu}{4(1-\mu)((1-\mu)^2)^{7/2}} - \right. \\ & \left. \frac{945q\mu}{16(1-\mu)^3((1-\mu)^2)^{5/2}} + \frac{4725Q\sqrt{\mu^2}}{16\mu^8} \right) A^2 + O[A]^{5/2}; \end{aligned}$$

$$\begin{aligned} a_4 = & \left(\frac{3\sqrt{3}q}{2((1-\mu)^2)^{5/2}} - \frac{3\sqrt{3}q\sqrt{(1-\mu)^2}\mu}{2(1-\mu)^6} + \frac{3\sqrt{3}Q\sqrt{\mu^2}}{2\mu^5} \right) \sqrt{A} + \\ & \frac{75\sqrt{3}qA^{3/2}}{4(-1+\mu)^5\sqrt{(-1+\mu)^2}} + \left(\frac{3}{2}q \left(\frac{\frac{9q}{Q} - \frac{9q}{Q\mu}}{((1-\mu)^2)^{5/2}} - \frac{90(1-q)}{Q((1-\mu)^2)^{5/2}\mu} \right) + \right. \\ & \frac{135q\sqrt{(1-\mu)^2}}{Q(1-\mu)^6} - \frac{243q^2\sqrt{(1-\mu)^2}}{2Q(1-\mu)^6} - \frac{27q^2\sqrt{(1-\mu)^2}\mu}{2Q(1-\mu)^6} + \frac{135\sqrt{\mu^2}}{\mu^7} - \\ & \left. \frac{135q\sqrt{\mu^2}}{\mu^7} - \frac{135\sqrt{\mu^2}}{\mu^6} + \frac{243q\sqrt{\mu^2}}{2\mu^6} + \frac{27q\sqrt{\mu^2}}{2\mu^5} \right) A^2 + O[A]^{5/2}; \end{aligned}$$

$$\begin{aligned} b_1 = & \left(-q\mu^5 - Q\sqrt{(-1 + \mu)^2}\sqrt{\mu^2} + 3Q\sqrt{(-1 + \mu)^2}\mu\sqrt{\mu^2} + \right. \\ & \left. Q\sqrt{(-1 + \mu)^2}\mu^3\sqrt{\mu^2} - 3Q\sqrt{(-1 + \mu)^2}(\mu^2)^{3/2} \right) / \left((-1 + \mu)^3\sqrt{(-1 + \mu)^2}\mu^5 \right) - \\ & \left(15 \left(-3q\mu^7 - 2Q\sqrt{(-1 + \mu)^2}\sqrt{\mu^2} + 10Q\sqrt{(-1 + \mu)^2}\mu\sqrt{\mu^2} + 20Q\sqrt{(-1 + \mu)^2} \right. \right. \\ & \left. \left. \mu^3\sqrt{\mu^2} - 10Q\sqrt{(-1 + \mu)^2}\mu^4\sqrt{\mu^2} + 2Q\sqrt{(-1 + \mu)^2}\mu^5\sqrt{\mu^2} - \right. \right. \\ & \left. \left. 20Q\sqrt{(-1 + \mu)^2}(\mu^2)^{3/2} \right) A \right) / \left(2 \left((-1 + \mu)^5\sqrt{(-1 + \mu)^2}\mu^7 \right) \right) + \\ & \left(30 \left(\sqrt{3}q\mu^6 - \sqrt{3}q^2\mu^6 - \sqrt{3}Q\sqrt{(-1 + \mu)^2}\sqrt{\mu^2} + \sqrt{3}qQ\sqrt{(-1 + \mu)^2}\sqrt{\mu^2} + \right. \right. \\ & 4\sqrt{3}Q\sqrt{(-1 + \mu)^2}\mu\sqrt{\mu^2} - 4\sqrt{3}qQ\sqrt{(-1 + \mu)^2}\mu\sqrt{\mu^2} + \\ & 4\sqrt{3}Q\sqrt{(-1 + \mu)^2}\mu^3\sqrt{\mu^2} - 4\sqrt{3}qQ\sqrt{(-1 + \mu)^2}\mu^3\sqrt{\mu^2} - \\ & \sqrt{3}Q\sqrt{(-1 + \mu)^2}\mu^4\sqrt{\mu^2} + \sqrt{3}qQ\sqrt{(-1 + \mu)^2}\mu^4\sqrt{\mu^2} - \\ & \left. \left. 6\sqrt{3}Q\sqrt{(-1 + \mu)^2}(\mu^2)^{3/2} + 6\sqrt{3}qQ\sqrt{(-1 + \mu)^2}(\mu^2)^{3/2} \right) \right. \\ & \left. A^{3/2} \right) / \left(Q(-1 + \mu)^3\sqrt{(-1 + \mu)^2}\mu^7 \right) - \\ & \left(945 \left(q\mu^9 - Q\sqrt{(-1 + \mu)^2}\sqrt{\mu^2} + 7Q\sqrt{(-1 + \mu)^2}\mu\sqrt{\mu^2} + 35Q\sqrt{(-1 + \mu)^2}\mu^3\sqrt{\mu^2} - \right. \right. \\ & 35Q\sqrt{(-1 + \mu)^2}\mu^4\sqrt{\mu^2} + 21Q\sqrt{(-1 + \mu)^2}\mu^5\sqrt{\mu^2} - \\ & \left. \left. 7Q\sqrt{(-1 + \mu)^2}\mu^6\sqrt{\mu^2} + Q\sqrt{(-1 + \mu)^2}\mu^7\sqrt{\mu^2} - 21Q\sqrt{(-1 + \mu)^2}(\mu^2)^{3/2} \right) \right) \end{aligned}$$

$$A^2) / \left(4 \left((-1 + \mu)^7 \sqrt{(-1 + \mu)^2 \mu^9} \right) \right) + O[A]^{5/2};$$

$$\begin{aligned} b_3 = & \left(-\frac{3q}{((1-\mu)^2)^{5/2}} + \frac{3q\mu}{((1-\mu)^2)^{5/2}} + \frac{15Q\sqrt{\mu^2}}{4\mu^7} - \frac{15Q(1-\mu)^2\sqrt{\mu^2}}{4\mu^7} - \frac{15Q\sqrt{\mu^2}}{2\mu^6} + \frac{3Q\sqrt{\mu^2}}{4\mu^5} \right) + \\ & \left(\frac{945q}{8((1-\mu)^2)^{7/2}} - \frac{45q}{8(1-\mu)^2((1-\mu)^2)^{5/2}} - \frac{45q\sqrt{(1-\mu)^2}}{4(1-\mu)^8} - \frac{945q\mu}{8((1-\mu)^2)^{7/2}} + \right. \\ & \left. \frac{45q\mu}{8(1-\mu)^2((1-\mu)^2)^{5/2}} + \frac{45q\sqrt{(1-\mu)^2}\mu}{4(1-\mu)^8} + \frac{135Q\sqrt{\mu^2}}{2\mu^7} \right) A - \\ & \left(90 \left(\sqrt{3}q\mu^6 - \sqrt{3}q^2\mu^6 - \sqrt{3}Q\sqrt{(-1+\mu)^2}\sqrt{\mu^2} + \sqrt{3}qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2} + \right. \right. \\ & 4\sqrt{3}Q\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} - 4\sqrt{3}qQ\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} + 4\sqrt{3}Q \\ & \sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} - 4\sqrt{3}qQ\sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} - \sqrt{3}Q\sqrt{(-1+\mu)^2} \\ & \mu^4\sqrt{\mu^2} + \sqrt{3}qQ\sqrt{(-1+\mu)^2}\mu^4\sqrt{\mu^2} - 6\sqrt{3}Q\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} + \\ & \left. \left. 6\sqrt{3}qQ\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} \right) A^{3/2} \right) / \left(Q(-1+\mu)^3\sqrt{(-1+\mu)^2}\mu^7 \right) + \\ & \left(4725 \left(q\mu^9 - Q\sqrt{(-1+\mu)^2}\sqrt{\mu^2} + 7Q\sqrt{(-1+\mu)^2}\mu\sqrt{\mu^2} + 35Q\sqrt{(-1+\mu)^2}\mu^3\sqrt{\mu^2} - \right. \right. \\ & 35Q\sqrt{(-1+\mu)^2}\mu^4\sqrt{\mu^2} + 21Q\sqrt{(-1+\mu)^2}\mu^5\sqrt{\mu^2} - \\ & \left. \left. 7Q\sqrt{(-1+\mu)^2}\mu^6\sqrt{\mu^2} + Q\sqrt{(-1+\mu)^2}\mu^7\sqrt{\mu^2} - 21Q\sqrt{(-1+\mu)^2}(\mu^2)^{3/2} \right) \right. \\ & \left. A^2 \right) / \left(4(-1+\mu)^7\sqrt{(-1+\mu)^2}\mu^9 \right) + O[A]^{5/2}; \end{aligned}$$

$$\begin{aligned} b_5 = & \left(\frac{3q}{8((1-\mu)^2)^{5/2}} - \frac{3q\mu}{8((1-\mu)^2)^{5/2}} + \frac{3Q\sqrt{\mu^2}}{8\mu^5} \right) + \left(-\frac{45q}{16(1-\mu)^2((1-\mu)^2)^{5/2}} - \frac{45q\sqrt{(1-\mu)^2}}{4(1-\mu)^8} + \right. \\ & \left. \frac{45q\mu}{16(1-\mu)^2((1-\mu)^2)^{5/2}} + \frac{45q\sqrt{(1-\mu)^2}\mu}{4(1-\mu)^8} - \frac{75Q\sqrt{\mu^2}}{8\mu^7} \right) A + \\ & \left(\frac{45\sqrt{3}(1-q)q}{4Q((1-\mu)^2)^{5/2}} - \frac{45\sqrt{3}(1-q)q}{4Q((1-\mu)^2)^{5/2}\mu} + \frac{45\sqrt{3}\sqrt{\mu^2}}{4\mu^7} - \frac{45\sqrt{3}q\sqrt{\mu^2}}{4\mu^7} - \right. \\ & \left. \frac{45\sqrt{3}\sqrt{\mu^2}}{4\mu^6} + \frac{45\sqrt{3}q\sqrt{\mu^2}}{4\mu^6} \right) A^{3/2} + \\ & \left(\frac{945q}{64(1-\mu)^4((1-\mu)^2)^{5/2}} + \frac{315q\sqrt{(1-\mu)^2}}{2(1-\mu)^{10}} - \frac{945q\mu}{64(1-\mu)^4((1-\mu)^2)^{5/2}} - \right. \\ & \left. \frac{315q\sqrt{(1-\mu)^2}\mu}{2(1-\mu)^{10}} - \frac{11025Q\sqrt{\mu^2}}{64\mu^9} \right) A^2 + O[A]^{5/2}; \end{aligned}$$

From above results D_2 upto first order in A is

$$\begin{aligned} D_2 = & \frac{9q\sqrt{(1-\mu)^2}\omega_1^2}{16(1-\mu)^6} - \frac{9q\sqrt{(1-\mu)^2}\mu\omega_1^2}{16(1-\mu)^6} + \frac{9Q\sqrt{\mu^2}\omega_1^2}{16\mu^5} + \\ & \frac{9q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\omega_3}{4(1-\mu)^6(-1+\mu)^4} + \frac{9Q^2\omega_3}{4(-1+\mu)^2\mu^6} - \frac{9Q^2\omega_3}{2(-1+\mu)^2\mu^5} + \frac{9Q^2\omega_3}{4(-1+\mu)^2\mu^4} - \\ & \frac{9q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\mu\omega_3}{2(1-\mu)^6(-1+\mu)^4} + \frac{9q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\mu^2\omega_3}{4(1-\mu)^6(-1+\mu)^4} - \end{aligned}$$

$$\begin{aligned}
& \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{4(1-\mu)^6(-1+\mu)^2} - \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{4(1-\mu)^6(-1+\mu)^2\mu^4} - \frac{9qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3}}{4(-1+\mu)^4\mu^4} + \\
& \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{(1-\mu)^6(-1+\mu)^2\mu^3} - \frac{27qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{2(1-\mu)^6(-1+\mu)^2\mu^2} + \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{(1-\mu)^6(-1+\mu)^2\mu} - \\
& \frac{3q\sqrt{(1-\mu)^2}\omega_1\omega_3}{(1-\mu)^6} + \frac{3q\sqrt{(1-\mu)^2}\mu\omega_1\omega_3}{(1-\mu)^6} - \frac{3Q\sqrt{\mu^2\omega_1\omega_3}}{\mu^5} - \\
& \frac{3q\sqrt{(-1+\mu)^2}\omega_3^2}{2(-1+\mu)^5} - \frac{3Q\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^3\mu^5} + \frac{9Q\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^3\mu^4} - \frac{9Q\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^3\mu^3} + \\
& \frac{3Q\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^3\mu^2} - \frac{5q^2\omega_3^2}{4(-1+\mu)^6\omega_1} - \frac{5Q^2\omega_3^2}{4(-1+\mu)^4\mu^6\omega_1} + \frac{5Q^2\omega_3^2}{(-1+\mu)^4\mu^5\omega_1} - \\
& \frac{15Q^2\omega_3^2}{2(-1+\mu)^4\mu^4\omega_1} + \frac{5Q^2\omega_3^2}{(-1+\mu)^4\mu^3\omega_1} - \frac{5Q^2\omega_3^2}{4(-1+\mu)^4\mu^2\omega_1} + \frac{5qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^6\mu^4\omega_1} - \\
& \frac{5qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3^2}}{(-1+\mu)^6\mu^3\omega_1} + \frac{5qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^6\mu^2\omega_1} - \frac{27q^2\omega_1^3}{16(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} - \\
& \frac{27Q^2\omega_1^3}{16\mu^6(\omega_1^2-4\omega_3^2)} + \frac{27q^2\mu\omega_1^3}{4(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} - \frac{81q^2\mu^2\omega_1^3}{8(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} + \\
& \frac{27q^2\mu^3\omega_1^3}{4(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} - \frac{27q^2\mu^4\omega_1^3}{16(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} + \frac{27qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{8(1-\mu)^6\mu^4(\omega_1^2-4\omega_3^2)} - \\
& \frac{27qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{4(1-\mu)^6\mu^3(\omega_1^2-4\omega_3^2)} + \frac{27qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{8(1-\mu)^6\mu^2(\omega_1^2-4\omega_3^2)} - \frac{9q^2\omega_1^2\omega_3}{16(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} - \\
& \frac{9Q^2\omega_1^2\omega_3}{16\mu^6(\omega_1^2-4\omega_3^2)} + \frac{9q^2\mu\omega_1^2\omega_3}{4(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} - \frac{27q^2\mu^2\omega_1^2\omega_3}{8(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} + \\
& \frac{9q^2\mu^3\omega_1^2\omega_3}{4(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} - \frac{9q^2\mu^4\omega_1^2\omega_3}{16(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} + \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{8(1-\mu)^6\mu^4(\omega_1^2-4\omega_3^2)} - \\
& \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{4(1-\mu)^6\mu^3(\omega_1^2-4\omega_3^2)} + \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{8(1-\mu)^6\mu^2(\omega_1^2-4\omega_3^2)} + \frac{9q^2\omega_1\omega_3^2}{2(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} + \\
& \frac{9Q^2\omega_1\omega_3^2}{2\mu^6(\omega_1^2-4\omega_3^2)} - \frac{18q^2\mu\omega_1\omega_3^2}{(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} + \frac{27q^2\mu^2\omega_1\omega_3^2}{(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} - \\
& \frac{18q^2\mu^3\omega_1\omega_3^2}{(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} + \frac{9q^2\mu^4\omega_1\omega_3^2}{2(1-\mu)^{10}(\omega_1^2-4\omega_3^2)} - \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^6\mu^4(\omega_1^2-4\omega_3^2)} + \\
& \frac{18qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^6\mu^3(\omega_1^2-4\omega_3^2)} - \frac{9qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^6\mu^2(\omega_1^2-4\omega_3^2)} + \\
& A \left(\frac{81q^2\omega_1}{4(1-\mu)^{10}} + \frac{81Q^2\omega_1}{4\mu^8} - \frac{81q^2\mu\omega_1}{2(1-\mu)^{10}} + \frac{81q^2\mu^2\omega_1}{4(1-\mu)^{10}} + \frac{81qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1}}{2(1-\mu)^6\mu^5} - \right. \\
& \frac{81qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1}}{2(1-\mu)^6\mu^4} - \frac{675q\sqrt{(1-\mu)^2}\omega_1^2}{32(1-\mu)^8} + \frac{675q\sqrt{(1-\mu)^2}\mu\omega_1^2}{32(1-\mu)^8} - \\
& \frac{225Q\sqrt{\mu^2\omega_1^2}}{16\mu^7} - \frac{135q^2\omega_1^2}{16(1-\mu)^{10}\omega_3} - \frac{135Q^2\omega_1^2}{16\mu^8\omega_3} + \frac{135q^2\mu\omega_1^2}{8(1-\mu)^{10}\omega_3} - \\
& \frac{135q^2\mu^2\omega_1^2}{16(1-\mu)^{10}\omega_3} - \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2}}{8(1-\mu)^6\mu^5\omega_3} + \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2}}{8(1-\mu)^6\mu^4\omega_3} - \\
& \frac{135q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\omega_3}{4(1-\mu)^6(-1+\mu)^6} - \frac{135q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\omega_3}{4(1-\mu)^8(-1+\mu)^4} - \\
& \frac{135q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\omega_3}{8(1-\mu)^7(-1+\mu)^4} - \frac{45Q^2\omega_3}{2(-1+\mu)^4\mu^8} - \frac{135Q^2\omega_3}{4(-1+\mu)^2\mu^8} + \frac{90Q^2\omega_3}{(-1+\mu)^4\mu^7} + \\
& \frac{135Q^2\omega_3}{2(-1+\mu)^3\mu^7} - \frac{135Q^2\omega_3}{(-1+\mu)^4\mu^6} - \frac{135Q^2\omega_3}{4(-1+\mu)^2\mu^6} + \frac{90Q^2\omega_3}{(-1+\mu)^4\mu^5} - \frac{45Q^2\omega_3}{2(-1+\mu)^4\mu^4} + \\
& \frac{135q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\mu\omega_3}{2(1-\mu)^6(-1+\mu)^6} + \frac{135q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\mu\omega_3}{2(1-\mu)^8(-1+\mu)^4} +
\end{aligned}$$

$$\begin{aligned}
& \frac{135q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\mu\omega_3}{8(1-\mu)^7(-1+\mu)^4} - \frac{135q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\mu^2\omega_3}{4(1-\mu)^6(-1+\mu)^6} - \\
& \frac{135q^2\sqrt{(1-\mu)^2}\sqrt{(-1+\mu)^2}\mu^2\omega_3}{4(1-\mu)^8(-1+\mu)^4} + \frac{45qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{2(1-\mu)^6(-1+\mu)^4} + \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{4(1-\mu)^8(-1+\mu)^2} + \frac{45qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{2(1-\mu)^6(-1+\mu)^4\mu^6} + \\
& \frac{135qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3}}{4(-1+\mu)^4\mu^6} - \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{(1-\mu)^6(-1+\mu)^4\mu^5} + \\
& \frac{675qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{2(1-\mu)^6(-1+\mu)^4\mu^4} + \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{4(1-\mu)^8(-1+\mu)^2\mu^4} + \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{8(1-\mu)^7(-1+\mu)^2\mu^4} + \frac{135qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3}}{4(-1+\mu)^6\mu^4} - \\
& \frac{450qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{(1-\mu)^6(-1+\mu)^4\mu^3} - \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{(1-\mu)^8(-1+\mu)^2\mu^3} - \\
& \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{8(1-\mu)^7(-1+\mu)^2\mu^3} + \frac{675qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{2(1-\mu)^6(-1+\mu)^4\mu^2} + \\
& \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{2(1-\mu)^8(-1+\mu)^2\mu^2} + \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{8(1-\mu)^7(-1+\mu)^2\mu^2} - \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{(1-\mu)^6(-1+\mu)^4\mu} - \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{(1-\mu)^8(-1+\mu)^2\mu} - \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3}}{8(1-\mu)^7(-1+\mu)^2\mu} + \frac{405q\sqrt{(1-\mu)^2}\omega_1\omega_3}{4(1-\mu)^8} - \frac{405q\sqrt{(1-\mu)^2}\mu\omega_1\omega_3}{4(1-\mu)^8} + \\
& \frac{135Q\sqrt{\mu^2\omega_1\omega_3}}{2\mu^7} + \frac{135q\sqrt{(-1+\mu)^2}\omega_3^2}{4(-1+\mu)^7} + \frac{45Q\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^5\mu^7} - \frac{225Q\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^5\mu^6} + \\
& \frac{225Q\sqrt{\mu^2\omega_3^2}}{(-1+\mu)^5\mu^5} - \frac{225Q\sqrt{\mu^2\omega_3^2}}{(-1+\mu)^5\mu^4} + \frac{225Q\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^5\mu^3} - \frac{45Q\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^5\mu^2} + \\
& \frac{75Q^2\omega_3^2}{(-1+\mu)^8\omega_1} + \frac{25Q^2\omega_3^2}{(-1+\mu)^6\mu^8\omega_1} - \frac{150Q^2\omega_3^2}{(-1+\mu)^6\mu^4\omega_1} + \frac{375Q^2\omega_3^2}{(-1+\mu)^6\mu^6\omega_1} - \\
& \frac{500Q^2\omega_3^2}{(-1+\mu)^6\mu^5\omega_1} + \frac{375Q^2\omega_3^2}{(-1+\mu)^6\mu^4\omega_1} - \frac{150Q^2\omega_3^2}{(-1+\mu)^6\mu^3\omega_1} + \frac{25Q^2\omega_3^2}{(-1+\mu)^6\mu^2\omega_1} - \\
& \frac{25qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3^2}}{(-1+\mu)^8\mu^6\omega_1} + \frac{100qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3^2}}{(-1+\mu)^8\mu^5\omega_1} - \\
& \frac{375qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^8\mu^4\omega_1} + \frac{175qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3^2}}{(-1+\mu)^8\mu^3\omega_1} - \\
& \frac{125qQ\sqrt{(-1+\mu)^2}\sqrt{\mu^2\omega_3^2}}{2(-1+\mu)^8\mu^2\omega_1} + \frac{405q^2\omega_1^3}{8(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} + \frac{405q^2\omega_3^3}{16(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} + \\
& \frac{405Q^2\omega_1^3}{8\mu^8(\omega_1^2-4\omega_3^2)} - \frac{405q^2\mu\omega_1^3}{2(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} - \frac{1215q^2\mu\omega_1^3}{16(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} + \\
& \frac{1215q^2\mu^2\omega_1^3}{4(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} + \frac{1215q^2\mu^2\omega_1^3}{16(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} - \frac{405q^2\mu^3\omega_1^3}{2(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} - \\
& \frac{405q^2\mu^3\omega_1^3}{16(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} + \frac{405q^2\mu^4\omega_1^3}{8(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} - \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{8(1-\mu)^6\mu^6(\omega_1^2-4\omega_3^2)} + \\
& \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{4(1-\mu)^6\mu^5(\omega_1^2-4\omega_3^2)} - \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{8(1-\mu)^8\mu^4(\omega_1^2-4\omega_3^2)} - \\
& \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{16(1-\mu)^7\mu^4(\omega_1^2-4\omega_3^2)} - \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{8(1-\mu)^6\mu^4(\omega_1^2-4\omega_3^2)} + \\
& \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{4(1-\mu)^8\mu^3(\omega_1^2-4\omega_3^2)} + \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{16(1-\mu)^7\mu^3(\omega_1^2-4\omega_3^2)} - \\
& \frac{405qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^3}}{8(1-\mu)^8\mu^2(\omega_1^2-4\omega_3^2)} + \frac{135q^2\omega_1^2\omega_3}{8(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} + \frac{135q^2\omega_1^2\omega_3}{16(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} +
\end{aligned}$$

$$\begin{aligned}
& \frac{135Q^2\omega_1^2\omega_3}{8\mu^8(\omega_1^2-4\omega_3^2)} - \frac{135q^2\mu\omega_1^2\omega_3}{2(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} - \frac{405q^2\mu\omega_1^2\omega_3}{16(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} + \\
& \frac{405q^2\mu^2\omega_1^2\omega_3}{4(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} + \frac{405q^2\mu^2\omega_1^2\omega_3}{16(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} - \frac{135q^2\mu^3\omega_1^2\omega_3}{2(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} - \\
& \frac{135q^2\mu^3\omega_1^2\omega_3}{16(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} + \frac{135q^2\mu^4\omega_1^2\omega_3}{8(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} - \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{8(1-\mu)^6\mu^6(\omega_1^2-4\omega_3^2)} + \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{4(1-\mu)^6\mu^5(\omega_1^2-4\omega_3^2)} - \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{8(1-\mu)^8\mu^4(\omega_1^2-4\omega_3^2)} - \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{16(1-\mu)^7\mu^4(\omega_1^2-4\omega_3^2)} - \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{8(1-\mu)^6\mu^4(\omega_1^2-4\omega_3^2)} + \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{4(1-\mu)^8\mu^3(\omega_1^2-4\omega_3^2)} + \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{16(1-\mu)^7\mu^3(\omega_1^2-4\omega_3^2)} - \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{8(1-\mu)^8\mu^2(\omega_1^2-4\omega_3^2)} - \frac{135q^2\omega_1\omega_3^2}{(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} - \frac{135q^2\omega_1\omega_3^2}{2(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} - \\
& \frac{135Q^2\omega_1\omega_3^2}{\mu^8(\omega_1^2-4\omega_3^2)} + \frac{540q^2\mu\omega_1\omega_3^2}{(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} + \frac{405q^2\mu\omega_1\omega_3^2}{2(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} - \\
& \frac{810q^2\mu^2\omega_1\omega_3^2}{(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} - \frac{405q^2\mu^2\omega_1\omega_3^2}{2(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} + \frac{540q^2\mu^3\omega_1\omega_3^2}{(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} + \\
& \frac{135q^2\mu^3\omega_1\omega_3^2}{2(1-\mu)^{11}(\omega_1^2-4\omega_3^2)} - \frac{135q^2\mu^4\omega_1\omega_3^2}{(1-\mu)^{12}(\omega_1^2-4\omega_3^2)} + \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^6\mu^6(\omega_1^2-4\omega_3^2)} - \\
& \frac{270qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^6\mu^5(\omega_1^2-4\omega_3^2)} + \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^8\mu^4(\omega_1^2-4\omega_3^2)} + \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{2(1-\mu)^7\mu^4(\omega_1^2-4\omega_3^2)} + \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^6\mu^4(\omega_1^2-4\omega_3^2)} - \\
& \frac{270qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^8\mu^3(\omega_1^2-4\omega_3^2)} - \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{2(1-\mu)^7\mu^3(\omega_1^2-4\omega_3^2)} + \\
& \frac{135qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^8\mu^2(\omega_1^2-4\omega_3^2)} + \frac{270q^2\omega_1^2\omega_3}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} + \frac{270Q^2\omega_1^2\omega_3}{\mu^8(-4\omega_1^2+\omega_3^2)} - \\
& \frac{540q^2\mu\omega_1^2\omega_3}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} + \frac{270q^2\mu^2\omega_1^2\omega_3}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} + \frac{540qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{(1-\mu)^6\mu^5(-4\omega_1^2+\omega_3^2)} - \\
& \frac{540qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1^2\omega_3}}{(1-\mu)^6\mu^4(-4\omega_1^2+\omega_3^2)} + \frac{54q^2\omega_1\omega_3^2}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} + \frac{54Q^2\omega_1\omega_3^2}{\mu^8(-4\omega_1^2+\omega_3^2)} - \\
& \frac{108q^2\mu\omega_1\omega_3^2}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} + \frac{54q^2\mu^2\omega_1\omega_3^2}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} + \frac{108qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^6\mu^5(-4\omega_1^2+\omega_3^2)} - \\
& \frac{108qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_1\omega_3^2}}{(1-\mu)^6\mu^4(-4\omega_1^2+\omega_3^2)} - \frac{54q^2\omega_3^3}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} - \\
& \frac{54Q^2\omega_3^3}{\mu^8(-4\omega_1^2+\omega_3^2)} + \frac{108q^2\mu\omega_3^3}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} - \frac{54q^2\mu^2\omega_3^3}{(1-\mu)^{10}(-4\omega_1^2+\omega_3^2)} - \\
& \frac{108qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3^3}}{(1-\mu)^6\mu^5(-4\omega_1^2+\omega_3^2)} + \frac{108qQ\sqrt{(1-\mu)^2}\sqrt{\mu^2\omega_3^3}}{(1-\mu)^6\mu^4(-4\omega_1^2+\omega_3^2)} \Bigg);
\end{aligned}$$

when $\mu \rightarrow 0.00025$, $A \rightarrow 0.00025$, $q \rightarrow 0.025$ and $Q > 0.00025$

$$\begin{aligned}
D_2 = & -3.59996 \times 10^{15} (-30720.2 + \omega_1) \omega_1 - \frac{8.64 \times 10^{18} \omega_1^2}{\omega_3} + \\
& (-1.12895 \times 10^{20} + 1.72798 \times 10^{16} \omega_1) \omega_3 + \frac{(2.55997 \times 10^{19} - 5.7599 \times 10^{15} \omega_1) \omega_3^2}{\omega_1} +
\end{aligned}$$

$$\frac{\omega_1^2(2.21184 \times 10^{20} \omega_1 + 5.5296 \times 10^{19} \omega_3)}{-4\omega_1^2 + 1\omega_3^2} + \frac{\omega_1^2(-4.31996 \times 10^{18} \omega_1 - 4.31996 \times 10^{18} \omega_3)}{-0.25\omega_1^2 + 1\omega_3^2}$$

when $\mu \rightarrow 0.00025$, $A \rightarrow 0.00025$, $q \rightarrow 0.025$, $Q \rightarrow 0.00025$ and $\omega_3 \rightarrow 1$ Using all the values of coefficients and algebraic manipulations we get the following expression for D_2 :

$$\begin{aligned} D_2 = & 5.76096 \times 10^{14} - \frac{3.2 \times 10^{14}}{\omega_1} - 1.92 \times 10^{11} \omega_1 + \\ & 3.6 \times 10^{10} \omega_1^2 + \frac{1.152 \times 10^{15} \omega_1}{-4 + \omega_1^2} - \frac{1.44 \times 10^{14} \omega_1^2}{-4 + \omega_1^2} - \frac{4.32 \times 10^{14} \omega_1^3}{-4 + \omega_1^2} + \\ & 0.00025 \left(-2.30423 \times 10^{23} + \frac{1.024 \times 10^{23}}{\omega_1} + 8.30131 \times 10^{22} \omega_1 - \right. \\ & 3.45744 \times 10^{22} \omega_1^2 - \frac{2.21184 \times 10^{23}}{1 - 4\omega_1^2} + \frac{2.21184 \times 10^{23} \omega_1}{1 - 4\omega_1^2} + \\ & \left. \frac{1.10592 \times 10^{24} \omega_1^2}{1 - 4\omega_1^2} - \frac{5.5296 \times 10^{23} \omega_1}{-4 + \omega_1^2} + \frac{6.912 \times 10^{22} \omega_1^2}{-4 + \omega_1^2} + \frac{2.0736 \times 10^{23} \omega_1^3}{-4 + \omega_1^2} \right) \end{aligned}$$

We have computed the value of D_2 numerically for various values of parameters when $\mu = 0.0025$ and $\omega_3 = 1$. The graphs are plotted D_2 versus ω_1 . In figures 1, plot I, II, III represent the respective values of $q_2 = 0.25, 0.50$, and 0.75 respectively when $q_1 = 0.25$ fixed and the vertical line shown in each graph is an asymptote. Effect of A_2 in which first three curves are for $A_2 = 0.0025$ and second three curves belong to $A_2 = 0.0050$. Similarly we have also obtained the effect of q_1 in figure 2 in which plot I and II correspond to $q_1 = 0.50$ and 0.75 respectively when $q_2 = 0.25$ fixed. Here first two curves are plotted for $A_2 = .0025$ and second two curves for $A_2 = .0050$.

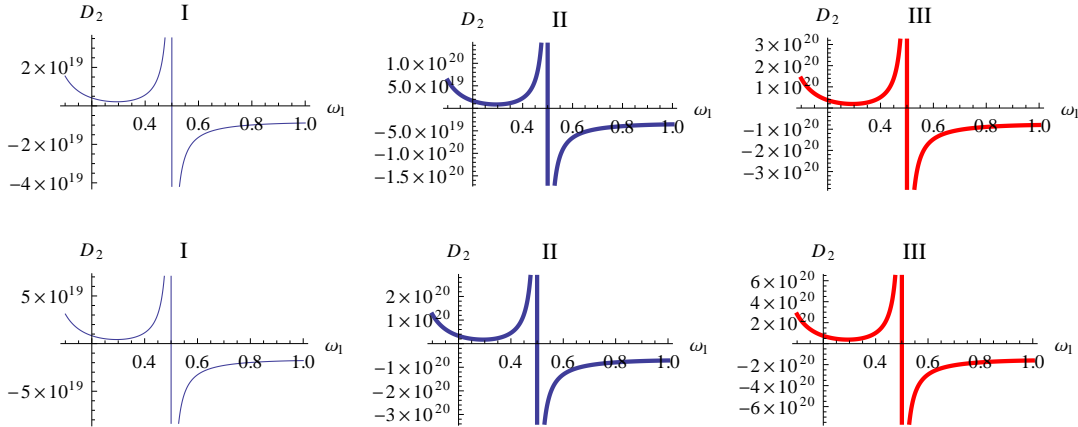


Fig. 1.— D_2 when $\mu \rightarrow 0.00025$, $A \rightarrow 0.00025$, $q_2 \rightarrow 0.25, 0.50$, and 0.75 , $Q \rightarrow 0.00025$ and $\omega_3 \rightarrow 1$

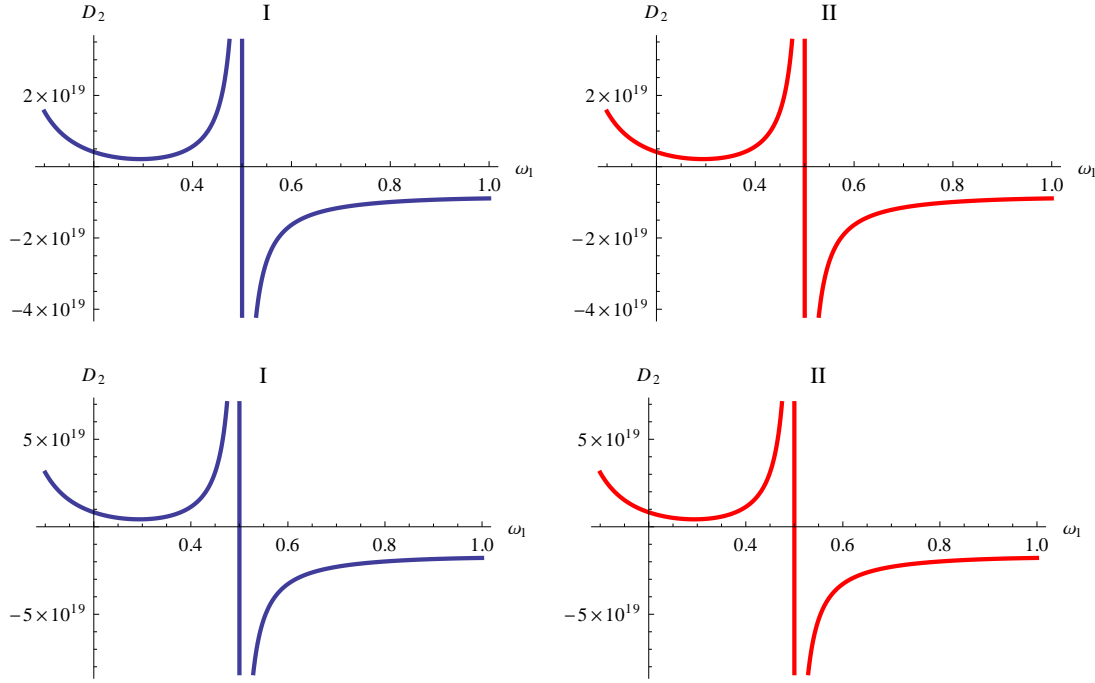


Fig. 2.— when $\mu \rightarrow 0.00025$, $A \rightarrow 0.00025$, $q_1 \rightarrow 0.50$ and 0.75 , $Q \rightarrow 0.00025$ and $\omega_3 \rightarrow 1$

3. Conclusion

It is evident from all the above figures that the curves are in rectangular hyperbolic forms with singularity at $\omega_1 = .50$ and we find that $D_2 \neq 0$ for various values of different parameters. Thus we conclude that, according to Arnold's theorem since $D_2 \neq 0$, out of plane equilibrium point L_6 is stable in non-linear sense.

Acknowledgements: We are thankful to D.S.T. Govt. of India, New Delhi for sanctioning a project SR/S4/MS:380/06,13/5/2008. We are also thankful to Dr Badam Singh Kushvah Department of Applied Mathematics, ISM, Dhanbad, India for valuable suggestions in preparing this manuscript during our visit to IUCAA, Pune.

REFERENCES

- Alfriend, K.T., Celest. Mech. 1, 351-359, 1970.
- Alfriend, K.T., Celest. Mech. 4, 60-77, 1971.
- Arnold, V.I., Soviet Math. Dokl. 2, 247-249, 1961.

- Coppola, V. T., & Rand, R. H. 1989, *Celestial Mechanics*, 45, 103
- Deprit, Andre., *Celestial Mechanics* 1, 12-30, 1969.
- Deprit, A. & Deprit- Bartholome, A. *Astron. J.* 72, 173-179,1967.
- Hori,G., *Publication Astronomical Society of Japan*, Vol 18, No 4, 287-296,1966.
- Hori,G., *Publ. Astron. Soc. Japan*, 19,229,1967.
- Jorba, À., & Villanueva, J. 1997, *Nonlinearity*, 10, 783
- Kushvah, B. S., Sharma, J. P., Ishwar, B. *Astrophysics and Space Science* 312, 279-293, 2007.
- Liu, J. C. 1985, *Celestial Mechanics*, 36, 89
- Maciejewski, A. J., & Gozdziwski, K. 1991, *Ap&SS*, 179, 1
- Markeev, A.P., *Appl. Math. Mech.* 33, 105-110, 1966.
- Meyer, K.R. & Schmidt, D.S., *Journal of Differential Equations* 62, 222-236,1986.
- Palacian,J. andYanguas,,P., *Non-linearity*,1021-1064,2000,U.K
- Shankaran,Sharma,J.P., and Ishwar,B.,*Astrophys.Space Sci.*, 332,115-119,2011.
- Sokolskii, A. G. 1978, *Celestial Mechanics*, 17, 373